Malaysian Journal of Mathematical Sciences 6(S): 111-121 (2012)
Special Edition of International Workshop on Mathematical Analysis (IWOMA)

On the Solution of Multipoint NBVP for Hyperbolic Equations with Integral Condition

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ABSTRACT

The Multipoint nonlocal boundary value problem for hyperbolic equations with integral condition is considered. Applying the operator approach, the stability estimates for the solution of this problem are obtained. Two applications of abstract results are given.

2000 MSC: 35L10, 35L20

Keywords: Hyperbolic partial differential equation, nonlocal boundary condition.

1. INTRODUCTION

It is well-known that hyperbolic equations with nonlocal integral conditions are widely used in plasma physics, thermoelasticity, chemical heterogeneity and etc. (see Mesloub *et al.* (1999); Pulkina (2000); Saadatmandi *et al.* (2006); Ramezani *et al.* (2007) and the references therein). Hyperbolic equations have been studied by many researchers (see Ashyralyev *et al.* (2001); Sabolevskii *et al.* (1983); Agarwal *et al.* (2004); Ashyralyev *et al.* (2009); Kostin (1909); Fattorini (1985); Piskarev *et al.* (1997) and Guidetti *et al.* (2004)).

In Ashyralyev *et al.* (2004), the nonlocal boundary-value problem

$$\begin{cases} \dfrac{d^2u(t)}{dt^2} + Au(t) = f(t) \quad (0 \le t \le 1), \\ u(0) = \alpha u(1) + \varphi, \quad u'(0) = \beta u'(1) + \psi \end{cases} \tag{1}$$

in a Hilbert space H with the self-adjoint positive definite operator A was studied. The stability estimates for the solutions of the nonlocal boundary-value problem (1) were established. In applications, the stability estimates for the solutions of the mixed type nonlocal boundary-value problems for hyperbolic equations were obtained. In the present paper, the multipoint nonlocal boundary value problem

$$\begin{cases} \dfrac{d^2u(t)}{dt^2} + Au(t) = f(t) \quad (0 \le t \le 1), \\ u(0) = \displaystyle\int_0^1 \alpha(\rho)u(\rho)d\rho + \varphi, \quad u_t(0) = \psi, \end{cases} \tag{2}$$

for the differential equation in a Hilbert space H with a self-adjoint positive definite operator A is considered. We are interested in studying the stability of solutions of problem (2) under the assumption

$$\int_0^1 |\alpha(\rho)| d\rho < 1. \tag{3}$$

2. THE MAIN THEOREM

A function $u(t)$ is called a solution of problem (2) if the following conditions are satisfied:

(i) $u(t)$ is twice continuously differentiable on the segment $[0,1]$.

(ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [0,1]$ and the function $Au(t)$ is continuous on the segment $[0,1]$.

(iii) $u(t)$ satisfies the equation, the nonlocal and the local boundary conditions (2).

Let H be a Hilbert space, A be a positive definite self-adjoint operator with $A \ge \delta I$, where $\delta > \delta_0 > 0$. Throughout this paper, $\{c(t),\ t \ge 0\}$ is a strongly continuous cosine operator-function defined by the formula

$$c(t) = \frac{e^{itA^{1/2}} + e^{itA^{1/2}}}{2}.$$

Then, from the definition of the sine operator-function $s(t)$

$$s(t)u = \int_0^t c(\rho)u \, d\rho$$

it follows that

$$s(t) = A^{1/2} \frac{e^{itA^{1/2}} - e^{itA^{1/2}}}{2i}.$$

For the theory of cosine operator-function we refer to Fattorini (1985) and Piskarev *et al.* (1997). Now, let us give some lemmas that will be needed below.

Lemma 1. The following estimates hold

$$\|c(t)\|_{H\to H} \le 1, \|A^{1/2}s(t)\|_{H\to H} \le 1. \tag{4}$$

Lemma 2. Suppose that assumption (3) holds. Then, the operator

$$I - \int_0^1 \alpha(\rho)c(\rho)d\rho$$

has an inverse

$$T = \left(I - \int_0^1 \alpha(\rho)c(\rho)d\rho \right)^{-1}$$

and the following estimate is satisfied

$$\|T\|_{H\to H} \le \frac{1}{I - \int_0^1 |\alpha(\rho)d\rho|}. \tag{5}$$

Proof. Applying the triangle inequality and estimate (4), we obtain

$$\left\| I - \int_0^1 \alpha(\rho)c(\rho)d\rho \right\|_{H\to H} \ge 1 - \int_0^1 |\alpha(\rho)| \|c(\rho)\|_{H\to H} d\rho \ge 1 - \int_0^1 |\alpha(\rho)| d\rho.$$

Estimate (5) follows from this estimate. Lemma 2 is proved.

Now, we shall obtain the formula for solution of problem (2). It is clear that (Fattorini (1985)) the initial value problem

$$\frac{d^2u}{dt^2}+Au(t)=f(t),\ \ 0<t<1,\ \ u(0)=u_0,u'(0)=u'_0 \tag{6}$$

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has a unique solution

$$u(t)=c(t)u_0+s(t)u'_0+\int_0^1 s(t-\lambda)f(\lambda)d\lambda\ . \tag{7}$$

Using (7) and the nonlocal boundary condition

$$u(0)=\int_0^1 \alpha(\rho)u(\rho)d\rho+\varphi$$

and the condition $u'(0)=\psi$, it can be written as follows

$$u(0)=\int_0^1 \alpha(\rho)c(\rho)d\rho u(0)+\int_0^1 \alpha(\rho)s(\rho)d\rho\psi+\int_0^1 \alpha(\rho)\int_0^\rho s(\rho-\lambda)f(\lambda)d\lambda\, d\rho+\varphi.$$

Then, from Lemma 2 it follows that

$$u(0)=T\left\{\int_0^1 \alpha(\rho)s(\rho)d\rho\psi+\int_0^1 \alpha(\rho)\int_0^\rho s(\rho-\lambda)f(\lambda)d\lambda\, d\rho+\varphi\right\}. \tag{8}$$

Consequently, if the function $f(t)$ is not only continuous, but also continuously differentiable on $[0,1]$, $\varphi\in D(A)$, $\psi\in D(A^{\frac{1}{2}})$, then the solution of problem (2) satisfy formulas (7) and (8).

Theorem 1. Suppose that $\varphi\in D(A)$, $\psi\in D(A^{\frac{1}{2}})$, $f(t)$ is a continuously differen-tiable function on $[0,1]$ and assuption (3) holds. Then, there is a unique solution of problem (2) and the stability inequalities

$$\max_{0\le t\le 1}\left\|u(t)\right\|_H\le M\left[\left\|\varphi\right\|_H+\left\|A^{-1/2}\psi\right\|_H+\max_{0\le t\le 1}\left\|A^{-1/2}f(t)\right\|_H\right], \tag{9}$$

$$\max_{0\le t\le 1}\left\|A^{1/2}u(t)\right\|_H \le M\left[\left\|A^{1/2}\varphi\right\|_H + \|\psi\|_H + \max_{0\le t\le 1}\|f(t)\|_H\right], \tag{10}$$

$$\max_{0\le t\le 1}\left\|\frac{d^2u(t)}{dt^2}\right\|_H + \max_{0\le t\le 1}\|Au(t)\|_H$$

$$\le M\left[\|A\varphi\|_H + \left\|A^{1/2}\psi\right\|_H + \|f(0)\|_H + \int_0^1 \|f'(t)\|_H\,dt\right] \tag{11}$$

are valid where M does not depend on $f(t),\ t\in[0,1],\ \varphi$ and ψ.

Proof. By Ashyralyev et al. (2001), we get

$$\max_{0\le t\le 1}\|u(t)\|_H \le \left[\|u(0)\|_H + \left\|A^{-1/2}u'(0)\right\|_H + \max_{0\le t\le 1}\left\|A^{-1/2}f(t)\right\|_H\right], \tag{12}$$

$$\max_{0\le t\le 1}\|u'(x)\|_H + \left\|A^{1/2}u(t)\right\|_H \tag{13}$$

$$\le M\left[\left\|A^{1/2}u(0)\right\|_H + \|Au'(0)\|_H + \max_{0\le t\le 1}\|f(t)\|_H\right],$$

$$\max_{0\le t\le 1}\left\|\frac{d^2u(t)}{dt^2}\right\|_H + \max_{0\le t\le 1}\|Au(t)\|_H + \max_{0\le t\le 1}\left\|A^{1/2}u(t)\right\|_H \tag{14}$$

$$\le M\left[\|Au(0)\|_H + \left\|A^{1/2}u(0)\right\|_H + \|f(0)\|_H + \int_0^1 \|f'(t)\|_H\,dt\right]$$

for the solution of problem (6).

The proof of Theorem 1 is based on estimates (12), (13) and (14) and the following estimates

$$\|u(0)\|_H \le M\left[\left\|A^{-1/2}u_0'\right\|_H + \max_{0\le t\le 1}\left\|A^{-1/2}f(t)\right\|_H + \|u_0\|\right], \tag{15}$$

$$\left\|A^{1/2}u(0)\right\|_H \le M\left[\left\|u_0'\right\|_H + \max_{0\le t\le 1}\left\|f(t)\right\|_H + \left\|A^{1/2}u_0\right\|\right], \tag{16}$$

$$\left\|Au(0)\right\|_H \le M\left[\left\|A^{1/2}u_0'\right\|_H + \max_{0\le t\le 1}\left\|A^{1/2}f(t)\right\|_H + \left\|Au_0\right\|\right]. \tag{17}$$

First, we prove (15). Using formula (8) and estimate (4), we obtain

$$\left\|u(0)\right\|_H \le \left\|T\right\|_{H\to H}\left\{\int_0^1 |\alpha(\rho)|\left\|A^{1/2}s(\rho)_0\right\|_{H\to H} d\rho \left\|A^{-1/2}\psi\right\|_H\right.$$
$$\left.+\int_0^1 |\alpha(\rho)|\int_0^\rho \left\|A^{1/2}s(\rho-\lambda)\right\|_{H\to H}\left\|A^{-1/2}f(\lambda)\right\|_H d\lambda\, d\rho + \left\|\varphi\right\|_H\right\}$$
$$\le M\left[\left\|A^{-1/2}\psi\right\|_H + \max_{0\le t\le 1}\left\|A^{-1/2}f(t)\right\|_H + \left\|\varphi\right\|_H\right].$$

Second, let us prove (16). From formula (8), estimate (4) it follows that

$$\left\|A^{1/2}u(0)\right\|_H \le \left\|T\right\|_{H\to H}\left\{\int_0^1 |\alpha(\rho)|\left\|A^{1/2}s(\rho)\right\|_{H\to H} d\rho \left\|\psi\right\|_H\right.$$
$$\left.+\int_0^1 |\alpha(\rho)|\int_0^\rho \left\|A^{1/2}s(\rho-\lambda)\right\|_{H\to H}\left\|f(\lambda)\right\|_H d\lambda\, d\rho + \left\|A^{1/2}\varphi\right\|_H\right\}$$
$$\le M\left[\left\|\psi\right\|_H + \max_{0\le t\le 1}\left\|f(t)\right\|_H + \left\|A^{1/2}\varphi\right\|_H\right].$$

Third, using formula (17), we get

$$Au(0) = T\left\{\int_0^1 \alpha(\rho)A^{1/2}s(\rho)d\rho A^{1/2}\psi\right.$$
$$\left.+\int_0^1 \alpha(\rho)\left(f(\rho) - c(\rho)f(0) - \int_0^\rho c(\rho-\lambda)f'(\lambda)d\lambda\right)d\rho + A\varphi\right\}.$$

It follows from estimate (4) that

$$
\begin{aligned}
\|Au(0)\|_H \le \|T\|_{H\to H} \Bigg\{ & \int_0^1 |\alpha(\rho)| \left\|A^{1/2}s(\rho)\right\|_{H\to H} d\rho \left\|A^{1/2}\psi\right\|_H \\
& + \int_0^1 |\alpha(\rho)| \Big(\|f(\rho)\|_H + \|c(\rho)\|_{H\to H} \|f(0)\|_H \\
& + \int_0^{\rho} \|c(\rho-\lambda)\|_{H\to H} \|f'(\lambda)\|_H \, d\lambda \Big) d\rho + \|A\varphi\|_H \Bigg\} \\
\le M \Big[& \left\|A^{1/2}\psi\right\|_H + \max_{0\le t\le 1} \left\|A^{1/2} f(t)\right\|_H + \|A\varphi\|_H \Big].
\end{aligned}
$$

Thus, Theorem 1 is proved.

3. APPLICATIONS

First, the mixed problem for hyperbolic equation

$$
\begin{cases}
u_{tt} - (a(x)u_x)_x + \sigma u = f(t,x), \ 0 < t < 1, \ 0 < x < 1, \\
u(0,x) = \int_0^1 \alpha(\rho) u(\rho,x) d\rho + \varphi(x), \quad u_t(0,x) = \psi(x), \ 0 \le x \le 1, \\
u(t,0) = u(t,1), \ u_x(t,0) = u_x(t,1), \ 0 \le t \le 1
\end{cases}
\tag{18}
$$

under assumption (3) is considered. The problem (18) has a unique smooth solution $u(t,x)$ for (3), the smooth functions $a(x) \ge a > 0$ $(x \in (0,1))$, $\varphi(x)$, $\psi(x)$ $(x \in [0,1])$ and $f(t,x)$ $(t, x \in [0,1])$, σ positive constant. This allows us to reduce mixed problem (18) to nonlocal boundary value problem (2) in Hilbert space $H = L_2[0,1]$ with a self-adjoint positive definite operator A^x defined by (18).

Theorem 2. For solutions of the mixed problem (18), we have the following stability inequalities

$$
\max_{0\le t\le 1} \|u_x(t,\cdot)\|_{L_2[0,1]} \le M \left[\max_{0\le t\le 1} \|f(t,\cdot)\|_{L_2[0,1]} + \|\varphi_x\|_{L_2[0,1]} + \|\psi\|_{L_2[0,1]} \right],
$$

$$\max_{0\le t\le 1}\left\|u_{xx}(t,\cdot)\right\|_{L_2[0,1]}+\max_{0\le t\le 1}\left\|u_{txt}(t,\cdot)\right\|_{L_2[0,1]}$$

$$\le M\left[\max_{0\le t\le 1}\left\|f_t(t,\cdot)\right\|_{L_2[0,1]}+\left\|f(0,\cdot)\right\|_{L_2[0,1]}+\left\|\varphi_{xx}\right\|_{L_2[0,1]}+\left\|\psi_x\right\|_{L_2[0,1]}\right]$$

where M does not depend on $\varphi(x)$, $\psi(x)$ and $f(t,x)$.

The proof of Theorem 2 is based on abstract Theorem 3 and the symmetry properties of the space operator generated by problem (18).

Second, let Ω be the unit open cube in the m-dimensional Euclidean space $\mathbb{R}^m$ with boundary S, $\overline{\Omega}=\Omega\cup S$. In $[0,1]\times\Omega$, let us consider the mixed boundary value problem for the multidimensional hyperbolic equation

$$\begin{cases}\dfrac{\partial^2 u(t,x)}{\partial t^2}-\sum\limits_{r=1}^{m}(a_r(x)u_{x_r}=f(t,x),\\ x=(x_1,\dots,x_m)\in\Omega,\quad 0<t<1,\\ u(0,x)=\displaystyle\int_0^1\alpha(\rho)u(\rho,x)d\rho+\varphi(x),\quad x\in\overline{\Omega},\\ u_t(0,x)=\psi(x),\ 0\le x\le 1,\ x\in\overline{\Omega},\\ u(t,x)=0,\ x\in S,\quad 0\le r\le m\end{cases}\tag{19}$$

under assumption (3). Here, $a_r(x)$, $(x\in\Omega)$, $\varphi(x)$, $\psi(x)$ $(x\in\overline{\Omega})$ and $f(t,x)$ $(t\in(0,1),\ x\in\Omega)$ are given smooth functions and $a_r(x)\ge a>0$.

Let us introduce the Hilbert space $L_2(\overline{\Omega})$ of the all square integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\left\|f\right\|_{L_2(\overline{\Omega})}=\left\{\int\cdots\int_{x\in\overline{\Omega}}\left|f(x)\right|^2dx_1\cdots dx_m\right\}^{\frac{1}{2}}.$$

Problem (19) has a unique smooth solution $u(t,x)$ for (3) and the smooth functions $\varphi(x)$, $\psi(x)$, $a_r(x)$ and $f(t,x)$. This enables us to reduce mixed problem (19) to nonlocal boundary value problem (2) in Hilbert space

$H = L_2(\overline{\Omega})$ with a self-adjoint positive definite operator A^x defined by (19).

Theorem 3. For the solutions of mixed problem (19), the following stability inequalities

$$\max_{0\le t\le 1}\sum_{r=1}^{m}\left\|u_{x_r}(t,\cdot)\right\|_{L_2(\overline{\Omega})} \le M\left[\max_{0\le t\le 1}\left\|f(t,\cdot)\right\|_{L_2(\overline{\Omega})} + \sum_{r=1}^{m}\left\|\varphi_{x_r}\right\|_{L_2(\overline{\Omega})} + \left\|\psi\right\|_{L_2(\overline{\Omega})}\right],$$

$$\max_{0\le t\le 1}\sum_{r=1}^{m}\left\|u_{x_r x_r}(t,\cdot)\right\|_{L_2(\overline{\Omega})} + \max_{0\le t\le 1}\left\|u_{tt}(t,\cdot)\right\|_{L_2(\overline{\Omega})}$$

$$\le M\left[\max_{0\le t\le 1}\left\|f_t(t,\cdot)\right\|_{L_2(\overline{\Omega})} + \left\|f(0,\cdot)\right\|_{L_2(\overline{\Omega})} + \sum_{r=1}^{m}\left\|\varphi_{x_r x_r}\right\|_{L_2(\overline{\Omega})} + \sum_{r=1}^{m}\left\|\psi_{x_r}\right\|_{L_2(\overline{\Omega})}\right]$$

hold, where M does not depend on $\varphi(x)$, $\psi(x)$ and $f(t,x)$.

The proof of Theorem 3 is based on the abstract Theorem 1, the symmetry properties of the operator A^x defined by formula (19) and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 4. (Sobolevskii (1995)). For the solutions of the elliptic differential problem

$$A^x u(x) = \omega(x), \quad x \in \Omega,$$
$$u(x) = 0, \quad x \in S,$$

the following coercivity inequality holds

$$\sum_{r=1}^{m}\left\|u_{x_r x_r}\right\|_{L_2(\overline{\Omega})} \le M\left\|\omega\right\|_{L_2(\overline{\Omega})}.$$

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